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On Characterization of Best Approximation with Certain Constraints

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The paper improves the characterization theorem of a best uniform approximation by a set of generalized polynomials having restricted ranges of derivatives obtained in an earlier paper and gives a characterization of a best approximation with certain constraints in the L_p norm ($1 \leqslant p < +\infty$). These results are applicable to many standard approximations with constraints. © 1998 Academic Press

1. INTRODUCTION

Assume $\mathscr{X} \subset [a, b]$ is a compact set containing at least $n+1$ points, $\Phi_n = \text{span}\{\varphi_1, ..., \varphi_n\}$ is an n-dimensional subspace of $L_p[a, b]$ with $1 \leqslant p \leqslant +\infty$, and for a fixed nonnegative integer k, the kth derivatives $\varphi_1^{(k)}, ..., \varphi_n^{(k)}$ are continuous. For $s = 0, 1, ..., k$, assume that $\{\varphi_1^{(s)}, ..., \varphi_n^{(s)}\}$ has a maximal linearly independent subset which is an extended. Chebyshev system of order r_s on $[a, b]$ (see the definition in [10, Chap. 1, Sect. 2], and write

$$K_s = \{q \in \Phi_n : l_s(x) \leqslant q^{(s)}(x) \leqslant u_s(x), x \in [a, b]\},$$

where l_s and u_s are extended real valued functions such that $-\infty \leqslant l_s(x) \leqslant u_s(x) \leqslant +\infty$. Let

$$K_S = \bigcap_{s=0}^{k} K_s.$$

With respect to uniform approximation (i.e., $p = +\infty$) by K_0, which is the set of generalized polynomials having restricted ranges, Taylor [2] (1969) got a characterization theorem of a best approximation under the hypothesis $l_0 < u_0$. The investigation by Shih [3] (1980) allows $l_0(x_i) = u_0(x_i)$ at a set of nodes $\{x_i\}$, but some strong conditions are required. Getting rid of Shih's strong conditions, the author [4] (1992) and Zhong [5] (1993) independently gave the characterization theorems in forms of

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convex hulls and alternation in the general case of $l_0(x) \leqslant u_0(x)$, which contains the special cases of approximation with interpolatory constraints, one-sided approximation, and copositive approximation. As we pointed out in [4], all the characterization theorems in [6], [7], and [8] are special cases of the case in [4]. However, the later result of Zhong [9] (1993) is not a special case of [4] because in order to apply it to the copositive case, $\{\varphi_1, ..., \varphi_n\}$ must be a Chebyshev system of order 2 while it is only required to be a Chebyshev system by [9].

Recently, we [1] got a characterization of a best uniform approximation by K_S, which has many special cases such as monotone approximation, coconvex approximation, multiple comonotone approximation, approximation with Hermite–Birkhoff interpolatory side conditions, and approximation by algebraic polynomials having bounded coefficients (if $0 \in [a, b]$), etc.

In this paper, we first improve the result of [1] and then give a characterization theorem of a best L_p $(1 \leqslant p < +\infty)$ approximation by the product of K_S and a so-called "local convex cone."

2. MAIN RESULTS

To introduce the main results of this paper, we need some notation.

For a fixed $q_0 \in K_s$, let

$$d(q_0^{(s)}(x), l_s) = \inf_{\xi \in [a, b]} \sqrt{(\xi - x)^2 + [l_s(\xi) - q_0^{(s)}(x)]^2},$$

and define $d(q_0^{(s)}(x), u_s)$ similarly. Write the set of all the nodes of K_s as

$$X_s^* = \{x \in [a, b] : d(q_0^{(s)}(x), l_s) = d(q_0^{(s)}(x), u_s) = 0\}.$$

If $x \in [a, b)$, by the use of

$$\lim_{\xi \to x+0} \frac{u_s(\xi) - q_0^{(s)}(\xi)}{|\xi - x|^{t-1}} = 0 \tag{1}$$

we define an integer-valued function $t_{s,1,1}(x)$ as follows:

$$t_{s,1,1}(x) = \begin{cases} 0, & \text{if } x \notin X_s^* \text{ and (1) does not hold for any positive integer } t, \\ 1, & \text{if } x \in X_s^* \text{ and (1) does not hold for any positive integer } t, \\ \tau, & \text{if there exists a positive integer } \tau < r_s \text{ such that (1) holds} \\ & \quad \text{for } t = \tau \text{ but not for } t = \tau + 1, \\ r_s + 1, & \text{if (1) holds for } t = r_s \text{ but not for any positive integer } t, \\ +\infty, & \text{if (1) holds for any positive integer } t. \end{cases}$$

Similarly, using

$$\lim_{\xi \to x+0} \frac{q_0^{(s)}(\xi) - l_s(\xi)}{|\xi - x|^{t-1}} = 0 \tag{2}$$

we define $t_{s,1,-1}(x)$. And substituting $x-0$ for $x+0$ in (1) and (2), we define $t_{s,-1,1}(x)$ and $t_{s,-1,-1}(x)$ respectively for $x \in (a, b]$.

Given $x \in [a, b]$, write

$$t_{\pm} = \max\{\min\{t_{s,1,1}(x), t_{s,1,-1}(x)\}, \min\{t_{s,-1,1}(x), t_{s,-1,-1}(x)\}\},$$
$$\omega = (-1)^{t_{\pm}},$$

and define

$$t_s(x) = \begin{cases} t_{\pm} + 1, & \text{if there exists a } \nu \text{ such that } t_{s,1,\nu}(x), t_{s,-1,-\omega\nu}(x) > t_{\pm}, \\ t_{\pm}, & \text{otherwise,} \end{cases}$$
$$T_s = \max_{x \in [a,b]} \{t_s(x)\}.$$

Similar to the explanation for $t(x)$ at the end of Section 3 of [4], where $t(x)$ coincides with $t_0(x)$ here, we see that under the condition of (4) below $t_s(x)$ is just the minimum of the orders of the zero x of $q_1 - q_2$ for all choices of $q_1, q_2 \in K_s$. So in fact $t_s(x)$ and T_s are independent of the choices of q_0, and hence we call $t_s(x)$ the *order of quasi-touch* of l_s and u_s at x, and T_s the *order of quasi-touch* of l_s and u_s on $[a, b]$.

In what follows we always assume that $q_0 \in K_s$ unless otherwise stated, and for each $s = 0, ..., k$,

$$\{q^{(s)} : q \in K_s\} \backslash \{q_0^{(s)}\} \neq \varnothing \tag{3}$$

and

$$\begin{cases} T_s \leqslant r_s, & \\ t_s(x) < r_s, & x \in X''_s, \end{cases} \tag{4}$$

where X''_s will be defined later.

Let

$$X'_s = \{x \in [a, b] \backslash X^*_s : d(q_0^{(s)}(x), l_s) \text{ or } d(q_0^{(s)}(x), u_s) = 0\},$$
$$\sigma_s(x) = \begin{cases} 1, & \text{if } x \in X'_s \text{ and } d(q_0^{(s)}(x), l_s) = 0, \\ -1, & \text{if } x \in X'_s \text{ and } d(q_0^{(s)}(x), u_s) = 0; \end{cases}$$
$$X''_s = \{x \in X^*_s : \text{there exist } \mu \text{ and } \nu \text{ such that } t_{s,\mu,\nu}(x) > t_s(x)\},$$
$$\sigma_s(x) = -\nu(-1)^{[(\mu-1)/2]\, t_s(x)}, \qquad \text{if } x \in X''_s \text{ and } t_{s,\mu,\nu}(x) > t_{s(x)};$$

and

$$\hat{x}=(\varphi_1(x), ..., \varphi_n(x)),$$

$$\hat{x}^{(s+t)}=(\varphi_1^{(s+t)}(x), ..., \varphi_n^{(s+t)}(x)),$$

$$N_s=\{\pm\hat{x}^{(s+t)} : t=0, 1, ..., t_s(x)-1, x\in X_s^*\}$$

$$\cup\{-\sigma_s(x)\,\hat{x}^{(s+t_s(x))} : x\in X_s'\cup X_s''\}.$$

Moreover, for $f\in C(\mathscr{X})$ or $f\in L_p[a, b]$ with $1\leqslant p<+\infty$, we write respectively

$$K_{q_0}^{\infty}=\{q\in\Phi_n : \|f-q\|_{\infty}<\|f-q_0\|_{\infty}\}$$

or

$$K_{q_0}^{p}=\{q\in\Phi_n : \|f-q\|_{p}<\|f-q_0\|_{p}\}.$$

And if $f\in C(\mathscr{X})$, we write

$$X=\{x\in\mathscr{X} : |f(x)-q_0(x)|=\|f-q_0\|_{\infty}\}$$

and

$$N_{q_0}=\{-\operatorname{sgn}[f(x)-q_0(x)]\,\hat{x} : x\in X\}.$$

By letting $q_1=\sum_{j=1}^n a_j\varphi_j$ and $q_2=\sum_{j=1}^n b_j\varphi_j$ be any elements of Φ_n, we define their inner product by $(q_1, q_2)=\sum_{j=1}^n a_jb_j$. For any subset A of the space Φ_n, we define

$$A^{\circ}=\{h\in\Phi_n : (q, h)\leqslant 0, \forall q\in A\}.$$

Let

$$\operatorname{cc}(A)=\left\{q : q=\sum_{j=1}^{m}\lambda_j q_j,\, q_j\in A,\, \lambda_j\geqslant 0,\, m \text{ is an arbitrary positive integer}\right\}$$

if $A\neq\varnothing$, and $\operatorname{cc}(A)=\{0\}$ if $A=\varnothing$. By $\overline{\operatorname{cc}}(A)$ we denote the closure of $\operatorname{cc}(A)$. And the *relative interior* of A in Φ_n, which we denote by $\operatorname{ri}(A)$, is defined as follows:

$$\operatorname{ri}(A)=\{q\in\operatorname{aff}(A) : \exists\delta>0,\, O(q, \delta)\cap\operatorname{aff}(A)\subset A\},$$

where

$$\operatorname{aff}(A) := \{\lambda_1 q_1 + \cdots + \lambda_m q_m \mid q_i \in A, \lambda_1 + \cdots + \lambda_m = 1\}$$

and $O(q, \delta)$ is the δ-neighborhood of q.

Now we can restate the main result of [1] as follows:

THEOREM A. *Assume that $f \in C(\mathscr{X}) \backslash K_S$, $K_{q_0}^{\infty} \neq \emptyset$. If*

$$\bigcap_{s=0}^{k} \operatorname{ri}(K_s) \neq \emptyset,$$

then q_0 is a best uniform approximation to f from K_S if and only if there exists a vector $h \in \operatorname{cc}(N_{q_0}) \backslash \{0\}$ such that

$$-h \in \overline{\operatorname{cc}}\left(\bigcup_{s=0}^{k} N_s\right).$$

Given a subscript set Λ, and for each $\lambda \in \Lambda$ a real number d_λ and a vector $h_\lambda \in \Phi_n \backslash \{0\}$, we say that

$$K_\Lambda := \{q \in \Phi_n : (q, h_\lambda) \leqslant d_\lambda, \lambda \in \Lambda\}$$

is a *local convex cone* at $q_0 \in K_\Lambda$ if there exists a $\delta > 0$ such that the δ-neighborhood of q_0 in Φ_n $O(q_0, \delta)$ satisfies

$$O(q_0, \delta) \subset \{q \in \Phi_n : (q, h_\lambda) \leqslant d_\lambda, \lambda \in \Lambda \backslash \Lambda'\},$$

where

$$\Lambda' = \{\lambda \in \Lambda : (q_0, h_\lambda) = d_\lambda\}.$$

Now, the first result of this paper is as follows:

THEOREM 1. *Assume that K_Λ is a local convex cone at $q_0 \in K := K_\Lambda \cap K_S$, $f \in C(\mathscr{X}) \backslash K$, $K_{q_0}^{\infty} \neq \emptyset$. If*

$$\operatorname{ri}(K_\Lambda) \cap \left[\bigcap_{s=0}^{k} \operatorname{ri}(K_s)\right] \neq \emptyset, \tag{5}$$

then q_0 is a best uniform approximation to f from K if and only if there exists a vector $h \in \operatorname{cc}(N_{q_0}) \backslash \{0\}$ such that

$$-h \in \overline{\operatorname{cc}}\left(\{h_\lambda : \lambda \in \Lambda'\} \cup \left(\bigcup_{s=0}^{k} N_s\right)\right). \tag{6}$$

And if in addition Λ' is a finite set, then (6) *can be substituted by*

$$-h \in \mathrm{cc}\left(\{h_\lambda : \lambda \in \Lambda'\} \cup \left(\bigcup_{s=0}^{k} N_s\right)\right).$$

Theorem 1 improves Theorem A in two respects. First, it allows us to add some linear constraints (i.e., $(q, h_\lambda) \leqslant d_\lambda$) to the coefficients of q in K. For example, the set of generalized polynomials with bounded coefficients $\{q = \sum_{i=1}^{n} a_i \varphi_i : \alpha_i \leqslant a_i \leqslant \beta_i,\ i = 1, ..., n\}$ is a special case of K_Λ. Second, when Λ' is a finite set, $\overline{\mathrm{cc}}(\bullet)$ in (6) can be rewritten as $\mathrm{cc}(\bullet)$, which is more precise in formulation and more valuable in applications.

The second result of the paper is a similar characterization theorem of a best approximation in the L_p norm $(1 \leqslant p < +\infty)$:

THEOREM 2. *Assume that K_Λ is a local convex cone at $q_0 \in K = K_\Lambda \cap K_S$, $f \in L_p \backslash K$, $1 \leqslant p < +\infty$, $K^p_{q_0} \neq \varnothing$, and* (5) *holds. If* mes $Z(f - q_0) = 0$ *when* $p = 1$, *where* mes $Z(f - q_0)$ *is the measure of the set*

$$Z(f - q_0) = \{x \in [a, b] : f(x) - q_0(x) = 0\},$$

then q_0 is a best L_p approximation to f from K if and only if

$$(c_1, ..., c_n) \in \overline{\mathrm{cc}}\left(\{h_\lambda : \lambda \in \Lambda'\} \cup \left(\bigcup_{s=0}^{k} N_s\right)\right), \tag{7}$$

where

$$c_i = \int_a^b \varphi_i\, |f - q_0|^{p-1}\, \mathrm{sgn}(f - q_0)\, dx, \qquad i = 1, ..., n.$$

And if in addition Λ' is a finite set, then (7) *can be substituted by*

$$(c_1, ..., c_n) \in \mathrm{cc}\left(\{h_\lambda : \lambda \in \Lambda'\} \cup \left(\bigcup_{s=0}^{k} N_s\right)\right).$$

3. PROOF OF THEOREM 1

If we apply Theorem (6.9.7) in [11] to the case being discussed here, then the theorem can be rewritten as

LEMMA A. *Assume that* $K \subset \Phi_n$ *is a closed convex set,* $q_0 \in K$. *If* $f \in C(\mathscr{X}) \backslash K$ *and* $K^{\infty}_{q_0} \neq \varnothing$ (*or* $f \in L_p[a, b] \backslash K$, $1 \leqslant p < +\infty$, *and* $K^{p}_{q_0} \neq \varnothing$), *then* q_0 *is a best approximation to* f *from* K *in uniform norm* (*or* L_p *norm*) *if and only if there exists a vector* $h \in (K^{\infty}_{q_0} - q_0)^{\circ} \backslash \{0\}$ (*or* $(K^{p}_{q_0} - q_0)^{\circ} \backslash \{0\}$) *such that* $-h \in (K - q_0)^{\circ}$.

Now we restate Proposition (6.9.2) in [11] and Lemmas 3 and 4 in [1] as follows:

LEMMA B. *If* $A \subset \Phi_n$, *then*

$$A^{\circ\circ} = \overline{\mathrm{cc}}(A).$$

And if A *is a convex compact set not containing the origin, then*

$$A^{\circ\circ} = \mathrm{cc}(A).$$

LEMMA C. *For* $s = 0, ..., k$, *we have*

$$(K_s - q_0)^{\circ} = \overline{\mathrm{cc}}(N_s).$$

LEMMA D. *If* $f \in C(\mathscr{X})$, $q_0 \in \Phi_n$, *and* $K^{\infty}_{q_0} \neq \varnothing$, *then*

$$(K^{\infty}_{q_0} - q_0)^{\circ} = \mathrm{cc}(N_{q_0}).$$

LEMMA 1. *Assume* C_i, $i = 0, 1, ..., m$, *are closed convex subsets of* Φ_n, $0 \in \bigcap_{i=0}^{m} C_i$ *and* $\bigcap_{i=0}^{m} \mathrm{ri}(C_i) \neq \varnothing$, *then*

$$\left(\bigcap_{i=0}^{m} C_i\right)^{\circ} = \mathrm{cc}\left(\bigcup_{i=0}^{m} C_i^{\circ}\right).$$

Proof. Since $(C_0)^{\circ} = \mathrm{cc}(C^{\circ})_0$, we can assume inductively

$$\left(\bigcap_{i=0}^{l-1} C_i\right)^{\circ} = \mathrm{cc}\left(\bigcap_{i=0}^{l-1} C_i^{\circ}\right).$$

We will now prove

$$\left(\bigcap_{i=0}^{l} C_i\right)^{\circ} = \mathrm{cc}\left(\bigcup_{i=0}^{l} C_i^{\circ}\right).$$

Take $g_0 \in \bigcap_{i=0}^{m} \mathrm{ri}(C_i)$. For $j=0, ..., l$, by $C_j^\circ \subset \overline{\mathrm{cc}}(\bigcup_{i=0}^{l} C_i^\circ)$, the definition of $(\bullet)^\circ$, and Lemma B we get

$$\left(\overline{\mathrm{cc}}\left(\bigcup_{i=0}^{l} C_i^\circ\right)\right)^\circ \subset C_j^{\circ\circ} = \overline{\mathrm{cc}}(C_j).$$

So for any $g \in (\overline{\mathrm{cc}}(\bigcup_{i=0}^{l} C_i^\circ))^\circ$, by the convexity of $\mathrm{cc}(C_j)$ we see that for any $\lambda \in (0, 1)$

$$g_\lambda := \lambda g + (1-\lambda)\, g_0 \in \mathrm{cc}(C_j), \qquad j=0, 1, ..., l.$$

Since $0 \in \bigcap_{i=0}^{m} C_i$, there exists an $\varepsilon > 0$ such that $\varepsilon g_\lambda \in \bigcap_{i=0}^{l} C_i$. So $g_\lambda \in \mathrm{cc}(\bigcap_{i=0}^{l} C_i)$ and hence $g \in \overline{\mathrm{cc}}(\bigcap_{i=0}^{l} C_i)$. So

$$\left(\overline{\mathrm{cc}}\left(\bigcup_{i=0}^{l} C_i^\circ\right)\right)^\circ \subset \overline{\mathrm{cc}}\left(\bigcap_{i=0}^{l} C_i\right).$$

On the other hand, for any $g \in \overline{\mathrm{cc}}(\bigcap_{i=0}^{l} C_i)$, based on Lemma B we have $g \in \overline{\mathrm{cc}}(C_j) = C_j^{\circ\circ}$, $j=0, 1, ..., l$. So by the definition of $(\bullet)^\circ$ we get $g \in (\overline{\mathrm{cc}}(\bigcup_{i=0}^{l} C_i^\circ))^\circ$. Then

$$\left(\overline{\mathrm{cc}}\left(\bigcup_{i=0}^{l} C_i^\circ\right)\right)^\circ = \overline{\mathrm{cc}}\left(\bigcap_{i=0}^{l} C_i\right).$$

Combined with Lemma B we get

$$\left(\bigcap_{i=0}^{l} C_i\right)^\circ = \left(\overline{\mathrm{cc}}\left(\bigcap_{i=0}^{l} C_i\right)\right)^\circ = \left(\overline{\mathrm{cc}}\left(\bigcap_{i=0}^{l} C_i^\circ\right)\right)^{\circ\circ}$$
$$= \overline{\mathrm{cc}}\left(\bigcup_{i=0}^{l} C_i^\circ\right).$$

Now to complete the proof it is sufficient to show

$$\overline{\mathrm{cc}}\left(\bigcup_{i=0}^{l} C_i^\circ\right) = \mathrm{cc}\left(\bigcup_{i=0}^{l} C_i^\circ\right).$$

Write $\Psi = \mathrm{span}(\bigcap_{i=0}^{l-1} C_i)$. For any $g \in \overline{\mathrm{cc}}(\bigcup_{i=0}^{l} C_i^\circ)$, there exist $h_j \in \mathrm{cc}(\bigcup_{i=0}^{l} C_i^\circ)$, $j=1, 2, ...,$ such that

$$h_j \to g \qquad (j \to \infty).$$

Let

$$h_j = h_{1j} + h_{2j} + h_{3j} + h_{4j},$$

where

$$h_{1j}+h_{2j}\in \mathrm{cc}\left(\bigcup_{i=0}^{l-1} C_i^\circ\right)=\left(\bigcap_{i=0}^{l-1} C_i\right)^\circ,$$
$$h_{3j}+h_{4j}\in \mathrm{cc}(C_l^\circ)=C_l^\circ, \tag{8}$$

$$\begin{cases} h_{1j}, h_{3j}\in \Psi+\mathrm{span}\ C_l, \\ h_{2j}, h_{4j}\perp \Psi+\mathrm{span}\ C_l. \end{cases} \tag{9}$$

From the boundedness of $\{h_j\}$ we see that $\{h_{2j}+h_{4j}\}$ is bounded. So there exists a subsequence of $\{h_{2j}+h_{4j}\}$ (we still denote it by $\{h_{2j}+h_{4j}\}$ for convenience) and a $g_2\perp \Psi+\mathrm{span}\ C_l$ such that when $j\to\infty$

$$h_{2j}+h_{4j}\to g_2\in C_l^\circ. \tag{10}$$

Since $(h_{2j}, \bar{g})=0$ for any $\bar{g}\in\Psi$, by (8) we have

$$h_{1j}=(h_{1j}+h_{2j})-h_{2j}\in\left(\bigcap_{i=0}^{l-1} C_i\right)^\circ. \tag{11}$$

Similarly

$$h_{3j}\in C_l^\circ. \tag{12}$$

Assume that $\{|h_{1j}|\}$ is unbounded, then $\{h_{1j}/|h_{1j}|\}$ has a subsequence which converges to an $h\neq 0$. And by the boundedness of $\{h_{1j}+h_{3j}\}$ we see that $\{h_{3j}/|h_{1j}|\}$ converges to $-h$. Thus by (9), (11), and (12)

$$\begin{cases} h\in(\Psi+\mathrm{span}\ C_l)\cap\left(\bigcap_{i=0}^{l-1} C_i\right)^\circ, \\ -h\in C_l^\circ. \end{cases} \tag{13}$$

For $g_0\in\bigcap_{i=0}^{m}\mathrm{ri}(C_i)$ and any $\bar{g}\in\Psi$ there exists an $\varepsilon>0$ such that $g_0\pm\varepsilon\bar{g}\in\bigcap_{i=0}^{l-1} C_i$. So $(g_0\pm\varepsilon\bar{g}, h)\leqslant 0$. Since (13) implies $(g_0, \pm h)\leqslant 0$, hence $(g_0, h)=0$, we have $(\bar{g}, h)=0$. Similarly, $(\bar{\bar{g}}, h)=0$ for any $\bar{\bar{g}}\in\mathrm{span}\ C_l$. Then $h\perp(\Psi+\mathrm{span}\ C_l)$ which contradicts (13). Now we see that $\{|h_{1j}|\}$ is bounded and hence $\{|h_{3j}|\}$ is bounded too. So by (11) and (12) there exist g_1 and g_3 such that

$$h_{1j}\to g_1\in\left(\bigcap_{i=0}^{l-1} C_i\right)^\circ, \qquad h_{3j}\to g_3\in C_l^\circ$$

(taking subsequences if necessary) when $j \to \infty$. Thus by (10) and the inductive assumption we have

$$g = g_1 + g_2 + g_3 \in \text{cc}\left(\bigcup_{i=0}^{l} C_i^\circ\right). \quad \blacksquare$$

LEMMA 2. *For each $s = 0, 1, ..., k$, if*

$$\delta_s = \inf\{|x_1 - x_2| : x_1, x_2 \in X_s^*, x_1 \neq x_2\},$$

then for any $x \in X_s^*$ there exists a positive $\delta_0 < \delta_s$ such that

$$(x, x+\delta_0] \cap X'_s = \emptyset \quad \text{or} \quad \sigma_s(\xi) = \sigma_s(x), \quad \xi \in (x, x+\delta_0] \cap X'_s, \tag{14}$$

and

$$[x-\delta_0, x) \cap X'_s = \emptyset \text{ or } \sigma_s(\xi) = (-1)^{t_s(x)} \sigma_s(x), \quad \xi \in [x-\delta_0, x) \cap X'_s. \tag{15}$$

Proof. Because for any $q \in K_s$ we have $q^{(s)}(x) = q_0^{(s)}(x)$, $x \in X_s^*$, by (3) and the definition of the extended Chebyshev system we conclude that X_s^* is a finite set and hence $\delta_s > 0$.

Assume $(x, x+\delta] \cap X'_s \neq \emptyset$ for any positive $\delta < \delta_s$.

If for any positive $\delta < \delta_s$ there exists ξ, $\eta \in (x, x+\delta] \cap X'_s$ such that $\sigma_s(\xi) = 1$, $\sigma_s(\eta) = -1$, then there exist two sequences $\{\xi_i\}$ and $\{\eta_i\}$ such that ξ_i, $\eta_i \to x+0$ $(i \to \infty)$ and

$$\begin{cases} d(q_0^{(s)}(\xi_i), l_s) = 0, \\ d(q_0^{(s)}(\xi_i), u_s) = 0, \end{cases} \qquad i = 1, 2, ...$$

So for any $q \in K_s$ we have

$$\begin{cases} q^{(s)}(\xi_i) - q_0^{(s)}(\xi_i) \geqslant 0, \\ q^{(s)}(\eta_i) - q_0^{(s)}(\eta_i) \leqslant 0, \end{cases} \qquad i = 1, 2, ...,$$

which implies that $q^{(s)} - q_0^{(s)} \equiv 0$ by the definition of the extended Chebyshev system. This contradicts the hypothesis of (3). Now we see that there exists a positive $\delta_0 < \delta_s$ such that $\sigma_s(\xi) \equiv \text{constant}$ for any $\xi \in (x, x+\delta_0] \cap X'_s$. Without loss of generality, we assume that the constant equals 1. So there exists a sequence $\{\xi_i\}$ with $\xi_i \to x+0$ $(i \to \infty)$ and $d(q_0^{(s)}(\xi_i), l_s) = 0$. Then by the definition we get directly $t_{s,1,-1}(x) = \infty$ and $\sigma_s(x) = 1$ which implies (14). The proof of (15) is similar. $\blacksquare$

LEMMA 3. *For* $0 \leqslant s \leqslant k$, $x \in X_s^*$, *if there is a positive* $\delta_0 < \delta_s$ *that satisfies* (14) *and* (15), *then*

$$(H - q_0)^\circ = \mathrm{cc}(M),$$

where

$$H = \{q \in \Phi_n : l_s(x) \leqslant q^{(s)}(x) \leqslant u_s(x),\ x \in [x - \delta_0, x + \delta_0]\}, \tag{16}$$

$$\begin{aligned} M = \{ &\pm \hat{x}^{(s+j)} : j = 0, 1, ..., t_s(x) - 1\} \\ &\cup \{-\sigma_s(\xi)\, \hat{\xi}^{(s + t_s(\xi))} : \xi \in [x - \delta_0, x + \delta_0] \cap (X'_s \cup X''_s)\}. \end{aligned} \tag{17}$$

Proof. By $\varphi_i^{(s)}[x_0, x_1, ..., x_j]$ we denoted the difference quotient of the jth order of $\varphi_i^{(s)}$. Write

$$[\widehat{x_0, x_1, ..., x_j}]^{(s)} = (\varphi_1^{(s)}[x_0, ..., x_j], ..., \varphi_n^{(s)}[x_0, ..., x_j]).$$

Based on the well-known property of the difference quotient with coalescent knots we have

$$[\underbrace{\widehat{x, ..., x}}_{j+1}]^{(s)} = \frac{1}{j!}\, \hat{x}^{(s+j)} \tag{18}$$

and

$$\frac{1}{x_j - x} \left\{ [\underbrace{\widehat{x, ..., x, x_j}}_{j-1}]^{(s)} - \frac{1}{(j-1)!}\, \hat{x}^{(s+j-1)} \right\} = [\underbrace{\widehat{x, ..., x, x_j}}_{j}]^{(s)}. \tag{19}$$

Write $t_s(x)$ as t for convenience. Since Lemma C implies $(H - q_0)^\circ = \overline{\mathrm{cc}}(M)$, it is sufficient to prove that $h \in \mathrm{cc}(M)$ if $h \in \overline{\mathrm{cc}}(M)$.

If $h = 0$, then $h \in \mathrm{cc}(M)$ clearly. Otherwise, there exist $h_i \neq 0$, $i = 1, 2, ...,$ such that $h_i \in \mathrm{cc}(M)$ and

$$h_i \to h \qquad (i \to \infty).$$

(i) Provided $x \in X''_s$, let $\sigma = \sigma_s(x)$. Since by the definition of t_s we have $t_s(\xi) = 0$ for any $\xi \in X'_s$, from the Carathéodory theorem we can write

$$h_i = \sum_{j=0}^{t} \theta_{ij} \hat{x}^{(s+j)} + \sum_{j=t+1}^{t+m_j} \theta_{ij} \hat{x}_{ij}^{(s)}, \tag{20}$$

where $0 \leqslant m_i \leqslant n+1$, $x_{ij} \in [x-\delta_0, x+\delta_0] \cap X'_s$, and

$$\begin{cases} -\sigma\theta_{it} \geqslant 0, \\ -\delta_s(x_{ij})\,\theta_{ij} > 0, \qquad j = t+1, ..., t+m_i. \end{cases} \tag{21}$$

Take a subsequence of $\{h_i\}$ if necessary (still denoted by $\{h_i\}$) such that m_i equals a constant m (clearly, $0 \leqslant m \leqslant n+1$); for each $j = t+1, ..., t+m$, $\sigma_s(x_{ij})$ $(i = 1, 2, ...)$ is a constant; and there exists an x_j such that $x_{ij} \to x_j$ $(i \to \infty)$. Then from (21), (14), and (15) we have

$$\begin{cases} -\sigma_s(x_j)\,\theta_{ij} > 0, & \text{if } j \in J_0 := \{j: x_j \neq x,\ j = t+1, ..., t+m\}, \\ -\sigma\theta_{ij} > 0, & \text{if } j \in J := \{j: x_j = x,\ j = t+1, ..., t+m\} \text{ and } x_{ij} > x, \\ -(-1)^t\,\sigma\theta_{ij} > 0, & \text{if } j \in J := \{j: x_j = x,\ j = t+1, ..., t+m\} \text{ and } x_{ij} < x. \end{cases} \tag{22}$$

Let

$$\begin{cases} \theta'_{ij} = \theta_{ij}, & j \in J_0 \text{ or } j = t, \\ \theta'_{il} = \theta_{il} + \dfrac{1}{l!} \sum\limits_{j \in J} \theta_{ij}(x_{ij}-x)^l, & l = 0, ..., t-1, \\ \theta'_{ij} = \theta_{ij}(x_{ij}-x)^t, & j \in J. \end{cases} \tag{23}$$

Since (19) implies

$$\begin{aligned} \hat{x}^{(s)}_{ij} - \sum_{l=0}^{t-1} (x_{ij}-x)^l \frac{1}{l!}\,\hat{x}^{(s+l)} &= (x_{ij}-x)[\widehat{x, x_{ij}}]^{(s)} - \sum_{l=1}^{t-1} (x_{ij}-x)^l \frac{1}{l!}\,\hat{x}^{(s+l)} \\ &= (x_{ij}-x)^2\,[\widehat{x, x, x_{ij}}]^{(s)} - \sum_{l=2}^{t-1} (x_{ij}-x)^l \frac{1}{l!}\,\hat{x}^{(s+l)} \\ &= \cdots \\ &= (x_{ij}-x)^t\,[\widehat{\underbrace{x, ..., x}_{t}, x_{ij}}]^{(s)}, \end{aligned}$$

we can rewrite h_i as

$$h_i = \sum_{j=0}^{t} \theta'_{ij}\hat{x}^{(s+j)} + \sum_{j \in J} \theta'_{ij}[\widehat{\underbrace{x, ..., x}_{t}, x_{ij}}]^{(s)} + \sum_{j \in J_0} \theta'_{ij}\hat{x}^{(s)}_{ij}.$$

Now we shall prove that the sequence $\{A_j\}$, $A_i := \max_{j=0, ..., t+m} |\theta'_{ij}|$, is bounded. In fact, otherwise $\{A_i\}$ (or its subsequence) satisfies $A_i \to +\infty$

$(i \to \infty)$; θ'_{ij}/A_i has a limit θ_j; and at least one of $\{\theta_j\}_{j=0}^{t+m}$ does not equal zero. Since $\lim_{i \to \infty} h_i/A_i = 0$, by (18) we see that zero equals

$$\sum_{j=0}^{t-1} \theta_j \hat{x}^{(s+j)} + \left(\theta_t + \frac{1}{t!} \sum_{j \in J} \theta_j\right) \hat{x}^{(s+t)} + \sum_{j \in J_0} \theta_j \hat{x}_j^{(s)}, \tag{24}$$

and (21)–(23) imply

$$\begin{cases} -\sigma\theta_t \geqslant 0, & \\ -\sigma\theta_j \geqslant 0, & j \in J, \\ -\sigma_s(x_j)\,\theta_j \geqslant 0, & j \in J_0. \end{cases} \tag{25}$$

Because the definition of extended Chebyshev system of order r_s and the hypothesis $t \leqslant r_s$ imply that $\{\hat{x}^{(s+j)}\}_{j=0}^{t-1}$ are linearly independent, therefore at least one of θ_j's $(j = t, ..., t+m)$ does not equal zero. Based on Lemma 5 of [4] (substituted Φ_n by span$\{\varphi_1^{(s)}, ..., \varphi_n^{(s)}\}$), there exists a $q \in K_s$ such that

$$\begin{cases} q^{(s+j)}(x) = 0, & j = 0, ..., t-1, \\ \sigma q^{(s+t)}(x) > 0, & \\ \sigma_s(x_j)\, q^{(s)}(x_j) > 0. & \end{cases}$$

So by (24) and (25) we have

$$\begin{aligned} 0 &= (0, q) \\ &= \sum_{j=0}^{t-1} \theta_j q^{(s+j)}(x) + \left[\theta_t + \frac{1}{t!} \sum_{j \in J} \theta_j\right] q^{(s+t)}(x) + \sum_{j \in J_0} \theta_j q^{(s)}(x_j) < 0, \end{aligned}$$

which is a contradiction. Thus A_i is bounded.

Now, if we write the limit of θ'_{ij} as θ_j, then $h = \lim_{i \to \infty} h_i$ still has the form of (24). And by (25) we have $h \in \mathrm{cc}\,(\mathrm{M})$.

(ii) If $x \notin X''_s$, then $[x - \delta_0, x + \delta_0] \cap X'_s = \varnothing$. So in (20) we have $m_i = 0$ and $\theta_{it} = 0$. Let $A_i = \max_{j=0, ..., t-1} |\theta_{ij}|$. Then from the linear independence of $\{\hat{x}^{(s+j)}\}_{j=0}^{t-1}$ it is not difficult to see that $\{A_i\}$ is bounded. So $h = \lim_{i \to \infty} h_i \in \mathrm{cc}(\mathrm{M})$. ▮

LEMMA 4. *For each* $s = 1, ..., k$,

$$(K_s - q_0)^\circ = \mathrm{cc}(N_s). \tag{26}$$

Proof. Assume that $X_s^* = \{x_1, ..., x_m\}$. By Lemma 2 there exists a positive $\delta_0 < \delta_s$ such that (14) and (15) hold for every $x \in X_s^*$. Write

$$H_0 = \{q \in \Phi_n : l_s(x) \leqslant q^{(s)}(x) \leqslant u_s(x),\ s \in [a, b] \backslash O(X_s^*, \delta_0)\},$$
$$M_0 = \{-\sigma_s(x)\, \hat{x}^{(s)} : x \in X_s' \backslash O(X_s^*, \delta_0)\}.$$

For each $i = 1, ..., m$, by H_i and M_i we denote respectively the sets of (16) and (17) with x substituted by x_i. Then

$$K_s = \bigcap_{i=0}^{m} H_i,$$

$$N_s = \bigcup_{i=0}^{m} M_i,$$

$$(H_i - q_0)^\circ = \mathrm{cc}(M_i), \qquad i = 1, ..., m.$$

Suppose

$$(H_0 - q_0)^\circ = \mathrm{cc}(M_0). \tag{27}$$

If by Lemma 5 in [4] we take a $q \in K_s$ such that

$$\begin{cases} q^{(s+j)}(x_i) = 0, & j = 0, 1, ..., t_s(x_i) - 1, \quad i = 1, ..., m, \\ \sigma_s(\xi)\, q^{(s + t_s(\xi))}(\xi) > 0, & \xi \in X_s' \cup X_s'', \end{cases} \tag{28}$$

then it is clear that

$$\tfrac{1}{2}(q - q_0) \in \bigcap_{i=0}^{m} \mathrm{ri}(H_i),$$

and by Lemma 1 we have

$$(K_s - q_0)^\circ = \left[\bigcap_{i=0}^{m} (H_i - q_0)\right]^\circ = \mathrm{cc}\left(\bigcup_{i=0}^{m} (H_i - q_0)^\circ\right) = \mathrm{cc}(N_s).$$

Now it is sufficient to prove (27). In fact, if $0 \notin \mathrm{co}(M_0)$, which denotes the convex hull of M_0, then from Lemma B we have

$$\overline{\mathrm{cc}}(\mathrm{co}(M_0)) = \mathrm{cc}(\mathrm{co}(M_0)).$$

So by Lemma C with K_s replaced by H_0 we get (27). On the other hand, it is impossible that $0 \in \mathrm{co}(M_0)$ because otherwise we have

$$\sum_{j=0}^{r} \lambda_j \sigma_s(\xi_j)\, \hat{\xi}_j^{(s)} = 0, \qquad \lambda_j < 0, \quad \xi_j \in X_s' \backslash O(X_s^*, \delta_0),$$

and hence for the q satisfying (28)

$$\sum_{j=0}^{r} \lambda_j \sigma_s(\xi_j)\, q^{(s)}(\xi_j) = (q, 0) = 0,$$

which contradicts the second inequality of (28). ∎

LEMMA 5. *If $K_\Lambda \subset \Phi_n$ is a local convex cone at $q_0 \in K_\Lambda$, then*

$$(K_\Lambda - q_0)^\circ = \overline{\mathrm{cc}}(\{h_\lambda \colon \lambda \in \Lambda'\}).$$

Proof. Since $[\overline{\mathrm{cc}}(A)]^\circ = A^\circ$, by Lemma B it is sufficient to prove that

$$\overline{\mathrm{cc}}(K_\Lambda - q_0) = [\overline{\mathrm{cc}}(\{h_\lambda \colon \lambda \in \Lambda'\})]^\circ.$$

Write

$$H_\lambda = \{q \in \Phi_n : (q, h_\lambda) \leqslant d_\lambda\}.$$

Assume $q \in \overline{\mathrm{cc}}(K_\Lambda - q_0)$. For any $\lambda \in \Lambda'$, it is clear that $q \in \overline{\mathrm{cc}}(H_\lambda - q_0)$ and $(q + q_0, h_\lambda) \leqslant d_\lambda$. So $(q, h_\lambda) \leqslant 0$, $\lambda \in \Lambda'$, and hence

$$q \in [\overline{\mathrm{cc}}(\{h_\lambda \colon \lambda \in \Lambda'\})]^\circ.$$

On the other hand, suppose $q \notin \overline{\mathrm{cc}}(K_\Lambda - q_0)$. By the definition of a local convex cone there exists a $\delta > 0$ such that

$$\delta q \in H_\lambda - q_0, \qquad \lambda \notin \Lambda'.$$

If

$$\delta q \in \overline{\mathrm{cc}}(H_\lambda - q_0), \qquad \lambda \in \Lambda',$$

then $\delta q \in K_\Lambda - q_0$ and $q \in \overline{\mathrm{cc}}(K_\Lambda - q_0)$, which contradicts the hypothesis. So there exists at least one $\lambda_0 \in \Lambda'$ such that $\delta q \notin \overline{\mathrm{cc}}(H_{\lambda_0} - q_0)$. So

$$(\delta q, h_{\lambda_0}) > 0,$$

which implies

$$q \notin [\overline{\mathrm{cc}}(\{h_\lambda \colon \lambda \in \Lambda'\})]^\circ. \quad ∎$$

The Proof of Theorem 1. By Lemmas 1, 4, and 5 we have

$$(K-q_0)^\circ=\overline{\mathrm{cc}}\left(\{h_\lambda: \lambda\in\Lambda'\}\cup\left(\bigcup_{s=0}^{k} N_s\right)\right).$$

And if in addition Λ' is a finite set, it is clear that

$$\overline{\mathrm{cc}}(\{h_\lambda: \lambda\in\Lambda'\})=\mathrm{cc}(\{h_\lambda: \lambda\in\Lambda'\}),$$

and hence

$$(K-q_0)^\circ=\mathrm{cc}\left(\{h_\lambda: \lambda\in\Lambda'\}\cup\left(\bigcup_{s=0}^{k} N_s\right)\right).$$

Combining this with Lemma A and Lemma D we get the conclusion of Theorem 1. ▮

4. PROOF OF THEOREM 2

LEMMA 6. *If* $f\in L_p$ $(1\leqslant p<+\infty)$, $q_0\in\Phi_n$, $K^p_{q_0}\neq\varnothing$, *and* mes $Z(f-q_0)=0$ *when* $p=1$, *then* $(c_1, ..., c_n)\neq 0$ *and*

$$(K^p_{q_0}-q_0)^\circ=\{-\eta(c_1, ..., c_n): \eta\geqslant 0\}, \tag{29}$$

where the c_i*'s are defined below* (7).

Proof. Write

$$h_0=(c_1, ..., c_n).$$

Based on the characterization theorem of a best L_p approximation by the linear subspace Φ_n (see [12, Theorems 3.3.1 and 3.3.2]), we see that if $h_0=0$ then q_0 is a best approximation to f from Φ_n, which contradicts the hypothesis of $K^p_{q_0}\neq\varnothing$. Thus $h_0\neq 0$.

Now, it is sufficient to prove

$$\overline{\mathrm{cc}}(K^p_{q_0}-q_0)=\{-h_0\}^\circ \tag{30}$$

because by Lemma B it follows from (30) that

$$(\overline{\mathrm{cc}}(K^p_{q_0}-q_0))^\circ=\overline{\mathrm{cc}}(\{-h_0\}),$$

which implies (29).

(i) For $q \in \overline{\mathrm{cc}}(K^p_{q_0} - q_0)$, we will prove $q \in \{-h_0\}^\circ$. Assume on the contrary that $(q, -h_0) > 0$; then there must be a $q_1 \in \mathrm{cc}(K^p_{q_0} - q_0)$ such that $(q_1, -h_0) > 0$. By the definition of h_0 we get

$$\int_a^b q_1 \, |f - q_0|^{p-1} \, \mathrm{sgn}(f - q_0) \, dx < 0. \tag{31}$$

It is easy to show that

$$\|f - q_0\|_p < \|f - q_0 - \delta q_1\|_p, \qquad \forall \delta > 0. \tag{32}$$

In fact, if $p = 1$, by (31) we have

$$\begin{aligned} \|f - q_0\|_1 &= \int_a^b (f - q_0 - \delta q_1) \, \mathrm{sgn}(f - q_0) \, dx + \delta \int_a^b q_1 \, \mathrm{sgn}(f - q_0) \, dx \\ &< \|f - q_0 - \delta q_1\|_1. \end{aligned}$$

If $p > 1$, then from the Hölder Inequality we have

$$\begin{aligned} \|f - q_0\|_p^p &= \int_a^b (f - q_0 - \delta q_1) \, |f - q_0|^{p-1} \, \mathrm{sgn}(f - q_0) \, dx \\ &\quad + \delta \int_a^b q_1 \, |f - q_0|^{p-1} \, \mathrm{sgn}(f - q_0) \, dx \\ &< \int_a^b |f - q_0 - \delta q_1| \, |f - q_0|^{p-1} \, dx \\ &\leqslant \|f - q_0 - \delta q_1\|_p \, \|f - q_0\|_p^{p-1}. \end{aligned}$$

And hence

$$\|f - q_0\|_p < \|f - q_0 - \delta q_1\|_p \qquad (p > 1).$$

Now we get (32) and hence $q_1 \notin \mathrm{cc}(K^p_{q_0} - q_0)$ which is a contradiction.

(ii) If $(q, -h_0) < 0$, then

$$\rho := (q, h_0) = \int_a^b q \, |f - q_0|^{p-1} \, \mathrm{sgn}(f - q_0) \, dx > 0. \tag{33}$$

Since $q \in L_p$ and $|f - q_0|^{p-1} \in L_{p'}$ (where $(1/p) + (1/p') = 1$), $|q| \, |f - q_0|^{p-1}$ is integrable on $[a, b]$. So by Lusin's Theorem and the property of absolute continuity of an integral there exists a closed subset F of $[a, b]\backslash Z(f-q_0)$ such that both $f-q_0$ and q are continuous on F, and the complementary set

$$E := [a, b]\backslash Z(f-q_0) - F$$

is so small that

$$\int_E |q|\, |f-q_0|^{p-1}\, dx < \frac{\rho}{4(2^{p-1}+1)}. \tag{34}$$

Clearly

$$\mu := \min_{x \in F} |f(x) - q_0(x)| > 0,$$

$$M := \max_{x \in F} \{ \max\{ |f(x) - q_0(x)|, |q(x)| \} \} < +\infty.$$

(a) Assume that $p = 1$. Let

$$0 < \delta < \frac{\mu}{2M}.$$

Then for $x \in F$ we have

$$\operatorname{sgn}[f(x) - q_0(x) - \delta q(x)] = \operatorname{sgn}[f(x) - q_0(x)]. \tag{35}$$

So by (34), (33), and the hypothesis of mes $Z(f-q_0) = 0$ we see

$$\begin{aligned} \|f - q_0 - \delta q\|_1 &= \int_E |f - q_0 - \delta q|\, dx + \int_F (f - q_0 - \delta q) \operatorname{sgn}(f - q_0)\, dx \\ &\leqslant \int_E |f - q_0|\, dx + \delta \int_E |q|\, dx + \int_F |f - q_0|\, dx \\ &\quad - \delta \int_F q \operatorname{sgn}(f - q_0)\, dx \\ &\leqslant \|f - q_0\|_1 + 2\delta \int_E |q|\, dx - \delta \int_{E+F} q \operatorname{sgn}(f - q_0)\, dx \\ &\leqslant \|f - q_0\|_1 + \frac{\delta\rho}{4} - \delta\rho < \|f - q_0\|_1. \end{aligned}$$

(b) Assume that $p>1$. Let

$$F_+=\{x\in F: f(x)-q_0(x)>0\},$$
$$F_-=\{x\in F: f(x)-q_0(x)<0\},$$
$$0<\delta<\min\left\{\frac{\mu}{2M}, \frac{\rho}{(p-1)(b-a)\,M^2(\mu/2)^{p-2}}, \frac{\rho}{(p-1)(b-a)\,M^2(2M)^{p-2}}, \left(\frac{\rho}{4\cdot 2^{p-1}\,\|q\|_p^p}\right)^{1/(p-1)}\right\}.$$

Then (35) holds for any $x\in F=F_+\cup F_-$. So by the Taylor Formula we have

$$|f-q_0-\delta q|^p = \begin{cases} (f-q_0)^p-\delta pq(f-q_0)^{p-1} \\ \quad +\frac{1}{2}\delta^2 p(p-1)\,q^2(f-q_0-\Delta q)^{p-2}, & x\in F_+, \\ (q_0-f)^p+\delta pq(q_0-f)^{p-1} \\ \quad +\frac{1}{2}\delta^2 p(p-1)\,q^2(-f+q_0+\Delta q)^{p-2}, & x\in F_-, \end{cases} \tag{36}$$

where $\Delta=\Delta(x)$ satisfies $0<\Delta(x)<\delta$. Considering $\delta<\mu/(2M)<1$, by the definition of μ and M we get

$$|f-q_0-\Delta q|^{p-2}<\begin{cases}(\mu/2)^{p-2}, & p<2, \\ (2M)^{p-2}, & p\geqslant 2,\end{cases} \qquad x\in F.$$

Then from the definition of δ it follows that

$$\frac{1}{2}\,\delta(p-1)\int_F q^2\,|f-q_0-\Delta q|^{p-2}\,dx$$
$$<\frac{1}{2}\,\delta(p-1)(b-a)\,M^2\max\{(\mu/2)^{p-2}, (2M)^{p-2}\}<\frac{\rho}{2}. \tag{37}$$

And for $x\in E$, by the Taylor Formula we have

$$\begin{aligned} |f-q_0-\delta q|^p &\leqslant [\,|f-q_0|+\delta\,|q|\,]^p \\ &=|f-q_0|^p+\delta p\,|q|(|f-q_0|+\Delta\,|q|)^{p-1} \\ &\leqslant |f-q_0|^p+\delta p\,2^{p-1}\,|q|\,|f-q_0|^{p-1}+\delta p(2\Delta)^{p-1}\,|q|^p, \end{aligned} \tag{38}$$

where $\Delta=\Delta(x)$ satisfies $0<\Delta(x)<\delta$.

Now, from (38), (36), (37), (34), (33), and the definition of δ we have

$$
\begin{aligned}
&\|f-q_0-\delta q\|_p^p \\
&\quad=\left(\int_E+\int_{Z(f-q_0)}+\int_F\right)|f-q_0-\delta q|^p\,dx \\
&\quad\leqslant\int_E[|f-q_0|^p+\delta p 2^{p-1}|q|\,|f-q_0|^{p-1}]\,dx \\
&\qquad+\left[\delta^p p 2^{p-1}\int_E|q|^p\,dx+\delta^p\int_{Z(f-q_0)}|q|^p\,dx\right] \\
&\qquad+\int_{F_+}\Big[|f-q_0|^p-\delta pq\,|f-q_0|^{p-1}\operatorname{sgn}(f-q_0) \\
&\qquad+\frac{1}{2}\delta^2 p(p-1)\,q^2\,|f-q_0-\Delta q|^{p-2}\Big]\,dx \\
&\qquad+\int_{F_-}\Big[|f-q_0|^p-\delta pq\,|f-q_0|^{p-1}\operatorname{sgn}(f-q_0) \\
&\qquad+\frac{1}{2}\delta^2 p(p-1)\,q^2\,|f-q_0-\Delta q|^{p-2}\Big]\,dx \\
&\quad\leqslant\|f-q_0\|_p^p+\delta p 2^{p-1}\int_E|q|\,|f-q_0|^{p-1}\,dx \\
&\qquad+\delta p\int_E q\,|f-q_0|^{p-1}\operatorname{sgn}(f-q_0)\,dx \\
&\qquad-\delta p\int_E q\,|f-q_0|^{p-1}\operatorname{sgn}|f-q_0|\,dx \\
&\qquad-\delta p\int_F q\,|f-q_0|^{p-1}\operatorname{sgn}(f-q_0)\,dx \\
&\qquad+\delta^p p 2^{p-1}\|q\|_p^p+\delta p\cdot\frac{1}{2}\delta(p-1)\int_F q^2\,|f-q_0-\Delta q|^{p-2}\,dx \\
&\quad\leqslant\|f-q_0\|_p^p+\delta p(2^{p-1}+1)\int_E|q|\,|f-q_0|^{p-1}\,dx \\
&\qquad-\delta p\int_a^b q\,|f-q_0|^{p-1}\operatorname{sgn}(f-q_0)\,dx+\delta p\,\delta^{p-1}2^{p-1}\|q\|_p^p+\delta p\,\frac{\rho}{2} \\
&\quad<\|f-q_0\|_p^p+\delta p\,\frac{\rho}{4}-\delta p\rho+\delta p\,\frac{\rho}{4}+\delta p\,\frac{\rho}{2} \\
&\quad=\|f-q_0\|_p^p.
\end{aligned}
$$

Based on (a) and (b), we see that if $(q, -h_0)<0$, then there exists a $\delta>0$ such that $q_0+\delta q\in K^p_{q_0}$, which means $q\in \mathrm{cc}(K^p_{q_0}-q_0)$. So if $(q, -h_0)\leqslant 0$ then $q\in\overline{\mathrm{cc}}(K^p_{q_0}-q_0)$, which is

$$\{-h_0\}^\circ\subset \mathrm{cc}(K^p_{q_0}-q_0).$$

Combining (i) with (ii) we obtain (30), and the lemma is established. ∎

Note. If we omit the condition that mes $Z(f-q_0)=0$ when $p=1$, then (29) may be false. A counterexample is as follows: Let $[a, b]=[-1, 1]$;

$$f(x)=\begin{cases}1, & x\geqslant 0,\\ 0, & x<0;\end{cases}\qquad n=2;\qquad \Phi_n=\mathrm{span}(1, x),\qquad \text{and}\qquad q_0(x)\equiv 0.$$

Then $K^1_{q_0}\neq\varnothing$ since $\|f-q_0\|_1=1$, and $\|f-((1/2)+(x/2))\|_1<1$. For any $q=a_1+a_2x$ with $a_1<0$, by drawing a diagram we can find that $\|f-q\|_1>1$. So

$$a_1\geqslant 0,\qquad \text{if}\quad q\in K^1_{q_0}.$$

Now let $q_1=(-1, 0)$. Then for any $q\in K^1_{q_0}$ we have $(q, q_1)\leqslant 0$. So

$$q_1\in (K^1_{q_0})^\circ=(K^1_{q_0}-q_0)^\circ.$$

But $q_1\notin\{-\eta(c_1, c_2):\eta\geqslant 0\}$ since $c_2=\int_{-1}^{1} x\,\mathrm{sgn}(f-q_0)\,dx=1/2$.

Proof of Theorem 2. The proof is similar to that of Theorem 1 in which one uses Lemma D instead of Lemma 6. ∎

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REFERENCES

1. S. S. Xu, Characterization of best uniform approximation with restricted ranges of derivatives, *Approx. Theory Appl.* **13** (1997), 37–48.
2. G. D. Taylor, Approximation by functions having restricted ranges, III, *J. Math. Anal. Appl.* **27** (1969), 241–248.
3. Y. K. Shih, Best approximation having restricted ranges with nodes, *J. Comput. Math.* **2** (1980), 124–132 [in Chinese].
4. S. S. Xu, A characterization of best approximation with restricted ranges, *J. Approx. Theory* **71** (1992), 193–212.
5. J. Zhong, Best restricted range approximation, *Numer. Funct. Anal. Optim.* **14** (1993), 179–194.

6. E. Passow and G. D. Taylor, An alternation theory for positive approximation, *J. Approx. Theory* **19** (1977), 123–134.
7. Y. K. Shih, An alternation theorem for copositive approximation, *Acta Math. Sinica* **24**, No. 3 (1981), 409–414 [in Chinese].
8. J. Zhong, On the characterization and strong uniqueness of best copositive approximation, *J. Comput. Math.* **10** (1988), 86–93 [in Chinese].
9. J. Zhong, Best copositive approximation, *J. Approx. Theory* **72** (1993), 210–233.
10. S. J. Karlin and W. J. Studden, "Tchebycheff Systems: with Applications in Analysis and Statistics," Interscience, New York, 1966.
11. P. J. Laurent, "Approximation et Optimisation," Collection Enseignement des Sciences, Vol. 13, Hermann, Paris, 1972.
12. N. P. Korneichuk, "Ekstremal'nye zadachi teorii priblizheniya," Nauka, Moscow, 1976.